Example:

Let S again be triangle with corners $(1,0,0)$, $(0,1,0)$, $(0,0,2)$

with parametrization $\Phi(u,v) = (u, v, 2-2u-2v)$,

$$\boxed{\begin{array}{l} 0 \leq u \leq 1 \\ 0 \leq v \leq 1-u \end{array}}$$

Let $F(x,y,z) = (-y, x, z)$

Then

$$\iint_{\Phi} F \cdot dS = \iint_{D} F(\Phi(u,v)) \cdot (T_u \times T_v)\, dv\, du$$

$$= \iint_{D} F(u, v, 2-2u-2v) \cdot (2,2,1)\, dv\, du$$

$$= \int_0^1 \int_0^{1-u} (-v, u, 2-2u-2v) \cdot (2,2,1)\, dv\, du$$

$$= \int_0^1 \int_0^{1-u} -2v + \cancel{2u} + 2 - \cancel{2u} - 2v\, dv\, du$$

$$= \int_0^1 2v - 2v^2 \Big|_0^{1-u} du = \int_0^1 2(1-u) - 2(1-u)^2\, du = \boxed{1/3}$$

Question: How does integral change when we change parametrization?

Example: Take same triangle S as before,

$$\Phi_2(u,v) = (v, u, 2-2u-2v)$$

$$0 \le u \le 1$$
$$0 \le v \le 1-u$$

check: • Φ_2 is also a parametrization of S

• $T_u = (0,1,-2)$, $T_v = (1,0,-2)$

$$T_u \times T_v = (-2,-2,-1)$$

Doing same calculation as on previous page we obtain with same $F(x,y,z) = (-y, x, z)$

$$\iint_{\Phi_2} F \cdot dS = \int_0^1 \int_0^{1-u} (-v, u, 2-2u-2v) \cdot (-2,-2,-1)\, dv\, du$$

$$= \boxed{-1/3}$$

Observe: $\iint_{\Phi_2} F \cdot dS = -1/3 = \iint_{\Phi} F \cdot dS$

This is as bad as it can get!

Theorem: Let Φ_1 and Φ_2 be parametrizations of the surface S

Let $\vec{n}_1$ and $\vec{n}_2$ be the normal vectors for Φ_1 and Φ_2

(recall: $\vec{n} = T_u \times T_v$)

let F be a vector field

$\Rightarrow$ (a) $\iint_{\Phi_1} F \cdot dS = \iint_{\Phi_2} F \cdot dS$ if $\vec{n}_1$ and $\vec{n}_2$ point to same side of S

(b) $\iint_{\Phi_1} F \cdot dS = -\iint_{\Phi_2} F \cdot dS$ if $\vec{n}_1$ and $\vec{n}_2$ point to opposite sides of S

path integral $\longleftrightarrow$ line integral

<u>path integral</u>: you integrate a FUNCTION

$$\int_c f \, ds$$

$$\int_c x^2+y^2 \, ds \qquad \text{if } f(x,y) = x^2+y^2$$

<u>Line integral</u> you integrate a VECTOR FIELD

notations:

$$\int_c F \cdot ds$$

or

$$\int_c f_1 dx + f_2 dy + f_3 dz$$

where $F(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z))$

path integral $\int_c f\,ds = \int_a^b f(c(t))\,\|c'(t)\|\,dt$

if $c:[a,b] \to \mathbb{R}^n$ $n=2,3$

line integral $\int_C F \cdot ds = \int_a^b F(c(t)) \cdot c'(t)\,dt$

example: $c(t) = (t, t^2)$ $0 \le t \le 3$

$F(x,y) = (yx, x)$

calculate $\int_C F \cdot ds$

aside: this is the same as computing

$$\int \underbrace{yx}_{f_1}\,dx + \underset{\overset{\shortparallel}{f_2}}{x}\,dy$$

$$F(x,y) = (xy,\ x)$$

$$c(t) = (t, t^2), \qquad 0 \leq t \leq 3$$

$$\int_C F \cdot ds = \int_0^3 F(c(t)) \cdot c'(t)\, dt$$

$$= \int_0^3 F(\underset{\overset{\shortparallel}{x}}{t}, \underset{\overset{\shortparallel}{y}}{t^2}) \cdot (1, 2t)\, dt$$

$$= \int_0^3 (t \cdot t^2, t) \cdot (1, 2t)\, dt$$

$$= \int_0^3 t^3 + 2t^2\, dt = \frac{t^4}{4} + \frac{2}{3} t^3 \Big|_0^3$$

$$= \frac{81}{4} + 2 \cdot 9$$

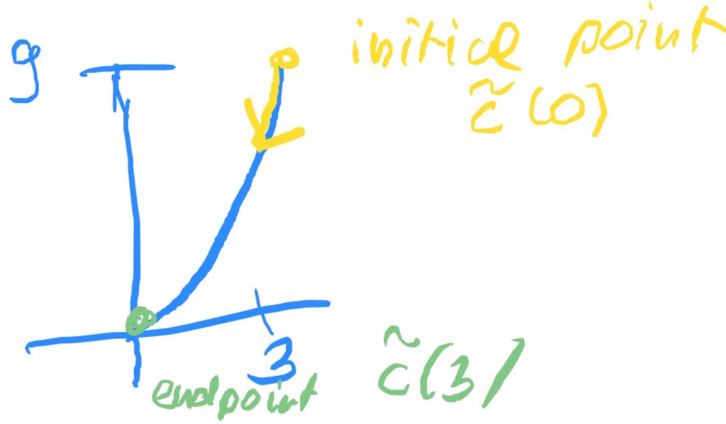
Properties of line integral

- independent of choice of $\underbrace{\text{parametrization}}_{\text{orientation preserving}}$

- If param orientation reversing
  $\Rightarrow$ integral changes to its negative

example: 1 $c(t) = (t, t^2)$

2 $\tilde{c}(t) = ((3-t), (3-t)^2)$



$$\Rightarrow \int_{\tilde{c}} F \cdot ds = - \int_{c} F \cdot ds$$

Special case:

If F is a gradient field

i.e. $F = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \leftarrow n=3$

$$\Rightarrow \int_C F \cdot ds = f(c(b)) - f(c(a))$$

$c : [a,b] \rightarrow \mathbb{R}^n \qquad n = 2, 3$

Example: Let $F(x,y,z) = \left(2x + ze^{xz}, 2y, 2z + xe^{xz}\right)$

$c : [0, 2\pi] \rightarrow \mathbb{R}^3 \qquad c(t) = \left(\cos^3 t, \sin^3 t, \left(\frac{t}{2\pi}\right)^5\right)$

Calculate $\int_C F \cdot ds$

- direct way: a pain!
- let's hope it is a gradient field

$$F(x,y,z) = \left(2x + ze^{xz},\ 2y,\ 2z + xe^{xz}\right)$$

try to find f

$$f = \int 2x + ze^{xz}\, dx$$

$$= x^2 + e^{xz} + c(y,z).$$

check 2nd coordinate

$$\boxed{2y = \frac{\partial f}{\partial y} = \frac{\partial c}{\partial y}} \Rightarrow c = y^2 + \tilde{c}(z)$$

integrate w.r.t. y

check z - coord.

$$2z + xe^{xz} = \frac{\partial f}{\partial z} = \partial/\partial z\left(x^2 + e^{xz} + y^2 + \tilde{c}(z)\right)$$

$$= 2x + xe^{xz} + \tilde{c}'(z)$$

$$\Rightarrow \boxed{\tilde{c}(z) = z^2}$$

$f(x,y,z) = x^2 + y^2 + z^2 + e^{xz}$

check $c(t) = \left(\cos^3 t, \sin^3 t, \left(\frac{t}{2\pi}\right)^5\right)$

$c(0) = (1,0,0)$ ← initial point

$c(2\pi) = (1,0,1)$ ← end point

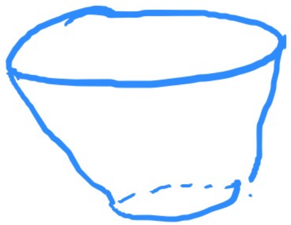
$\Rightarrow \int_c F \cdot ds = f(1,0,1) - f(1,0,0)$

$= \cancel{1} + \cancel{1} + e^1 - \cancel{1} - 0 - 0 - \cancel{e^0}$

$= \boxed{e}$

7.5 Problem 19

Find average value of $f(x,y,z) = x+z^2$ on truncated cone $\boxed{\begin{matrix} z^2 = x^2+y^2 \\ 1 \le z \le 4 \end{matrix}}$

S $\rightarrow$ 

Sol. $av(f) = \dfrac{\iint_S x+z^2 \, dS}{\iint_S 1 \, dS}$

$$z^2 = x^2 + y^2$$

$$1 \leq z \leq 4$$

$$\Rightarrow \quad z = \sqrt{x^2+y^2}$$

$$1 \leq \sqrt{x^2+y^2} \leq 4$$

use polar coordinates to parametrize S

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = \sqrt{x^2+y^2} = \sqrt{r^2(\sin^2\theta + \cos^2\theta)}$$

$$= r$$

$$\Rightarrow \quad \boxed{\begin{array}{l} \Phi(r,\theta) = (r\cos\theta,\ r\sin\theta,\ r) \\ 1 \leq r \leq 4 \quad (\text{translate } 1 \leq z \leq 4) \\ 0 \leq \theta \leq 2\pi \end{array}}$$

Calculating averages:

S given surface

$f: \mathbb{R}^3 \to \mathbb{R}$

$$\text{average of } f \text{ on } S = \frac{\iint_S f \, dS}{\text{area}(S)}$$

$$= \frac{\iint_S f \, dS}{\iint_S 1 \, dS}$$

$$\Phi(r,\theta) = (\overset{x}{\overset{\shortparallel}{r\cos\theta}}, \overset{y}{\overset{\shortparallel}{r\sin\theta}}, \overset{z}{\overset{\shortparallel}{r}})$$

$$T_r = \frac{\partial \Phi}{\partial r} = (\cos\theta, \sin\theta, 1)$$

$$T_\theta = [-r\sin\theta, r\cos\theta, 0]$$

need to calculate $\|T_r \times T_\theta\|$

$f(x,y,z) = x + z^2$

$$\boxed{av(f) = \frac{\int_0^{2\pi}\int_1^4 f(r\cos\theta, r\sin\theta, r)\,\|T_r \times T_\theta\|\,dr\,d\theta}{\int_0^{2\pi}\int_1^4 1 \cdot \|T_r \times T_\theta\|\,dr\,d\theta}}$$